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Empirical Logics

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"To what extent is logic empirical?" is a question that has been often discussed in connection with the studies about the foundations of quantum theory. Today we are facing not only a variety of logics, but even a variety of quantum logics. Hence, the original question seems to have turned to the new one: to what extent is it reasonable to look for the "right quantum logic"?

INTRODUCTION

"To what extent is logic empirical?" is a question that has been deeply discussed in the last 30 years, very often in connection with the studies about the logical foundations of quantum theory (QT). At the very beginning of this discussion, the claim that the choice of "the right logic" to be used in a given theoretical situation may depend also on experimental data appeared to be a kind of extremistic view, in contrast with a leading philosophical tradition according to which a characteristic feature of logic should be its absolute independence from any content.

These days, an empirical position in logic is generally no longer regarded as a "daring heresy." At the same time, we are facing today not only a variety of logics, but even a variety of quantum logics. As a consequence, the original question seems to have turned to the new one: to what extent is it reasonable to look for "the right quantum logic"?

The problem of how a physical theory can determine a logic has been deeply analyzed in the "renaissance period" of the logicoalgebraic approaches to QT after the appearence of Mackey's (1963) *Mathematical Foundations of Quantum Mechanics.* As is well known, the basic common assumption in these approaches appears quite natural: any physical theory T determines a collection of state-event systems $\langle S, E\rangle$, where S contains

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the *states* that a physical system described by the theory may assume and E contains the *events* that may occur to our system. Such state-event systems, in turn, may be interpreted as particular semantic structures that may give rise to different logics.

This construction involves at least two degrees of freedom:

1. The structural conditions required for our state-event systems may be not uniquely determined by the theory.

2. Different logics may be associated, according to different methods, to one and the same collection of state-event systems.

The same notion of "logic" has been used in this field somewhat ambiguously. According to a way of speaking that is more common in mathematical physics, a logic is often identified with a particular abstract structure whose operations admit to be interpreted as logical connectives. Let us think of the use of the term "quantum logic" for a σ-orthomodular poset.

In the logical tradition, instead, logics are described as more complex objects. Generally, a logic **L** can be determined as a pair consisting of a *proof-theoretic* part and of a *model-theoretic* (or *semantic*) part. The key notions are, respectively:

1. A proof-theoretic consequence relation:

$$T \vdash \alpha$$

(the sentence α is *provable* from the set of sentences T)

2. A model-theoretic consequence relation:

$$T \models \alpha$$

(the sentence α is a *semantic* consequence of the set of sentences T).

Different consequence relations may turn out to be equivalent. In such situations, one says that they *characterize* the same logic. A logic **L** is *axiomatizable* when **L** admits a proof-theoretic consequence relation $\vdash$, where the notion of *proof* is decidable. Further, **L** is *decidable* when the relation $\{\alpha\} \vdash \beta$ is decidable.

The *logical truths* of **L** are the sentences that are semantic consequences of the empty set.

1. STATE-EVENT SYSTEMS

Let us first discuss our first degree of freedom, concerning the state-event system.[2] According to a minimal interpretation, states may be

[2]For a general exposition see Beltrametti and Cassinelli (1981) and Pták and Pulmannová (1991). I will not consider the many terminological differences that can be found in the literature. In most cases, translations are obvious.

regarded as *pieces of information* about physical systems or objects (a kind of *individual concepts* in Leibniz's sense; whereas *pure states* correspond, to a certain extent, to what Leibniz called *complete concepts*). At the same time, events represent properties that may be verified by the objects under investigation. The general notion of property is far from being sharply characterized both in the logical and in the philosophical tradition. In the case of our state-event systems, one may adhere to an empirical interpretation of properties (as has been developed, for instance, in the work of Foulis and Randall, and in the operational approach to QT). Properties are supposed to be generated by means of logical tools (including abstraction) on the basis of a set of primitive *questions* that can be triggered by a measurement apparatus.

Let $u, v, w, \ldots$ represent elements of S, while $a, b, c, \ldots$ are elements of E. The minimal conditions that are required (and that, usually, are not an object of controversy) are the following:

1. $\forall u\ \forall a$: $u(a) \in [0, 1]$ (any state associates a probability-value to any event).
2. *Weak extensionality*:

$$\forall a, b: \quad \forall u\ [u(a) = u(b)] \Rightarrow a = b$$

$$\forall u, v: \quad \forall a\ [u(a) = v(a)] \Rightarrow u = v$$

In other words: events that are probabilistically indiscernible are identified. Similarly for states.

3. E is closed under a weak complement operation $^\perp$ such that

$$\forall a\ \forall u: \quad u(a^\perp) = 1 - u(a)$$

4. E contains a *certain event* $\mathbf{1}$ such that

$$\forall u: \quad u(\mathbf{1}) = 1$$

Let $\mathbf{0} := \mathbf{1}^\perp$ be the *impossible* event.

S permits us to define an order relation on E:

Definition 1.1:

$$a \leq b \Leftrightarrow \forall u\ [u(a) \leq u(b)]$$

Definition 1.2. Orthogonality:

$$a \perp b \Leftrightarrow a \leq b^\perp$$

One can prove:

Lemma 1.1. The structure $\langle E, \leq, ^\perp, \mathbf{1}, \mathbf{0} \rangle$ is an *involutive bounded regular poset*.

In other words:
(a) $\leq$ is a partial order with maximum $\mathbf{1}$ and minimum $\mathbf{0}$.
(b) $^{\perp}$ is an involution:
(ib) $a = a^{\perp\perp}$.
(iib) $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$.
(c) $a \perp a$ and $b \perp b \Rightarrow a \perp b$ (*regularity*).

5. A partial sum $\oplus$ is defined on E:
(a) $a \perp b \Rightarrow a \oplus b \in E$.
(b) $\forall u\ [u(a \oplus b) = u(a) + u(b)]$.

This represents a kind of minimal structure that it seems reasonable to require for a state-event system.

Further conditions that may strengthen $\langle S, E \rangle$ are the following:

5*. *Additivity*:

$$a \perp b \Rightarrow a \oplus b = \sup(a, b)$$

(the sum of two orthogonal events is their supremum).

5**. *σ-Additivity:* Let $\{a_i\}$ be a countable sequence of pairwise orthogonal events:
(a) $\sup\{a_i\} \in E$.
(b) $\forall u$: $u(\sup\{a_i\}) = \sum_i u(a_i)$.

6. The *noncontradiction principle*:

$$a \perp a \Rightarrow a = \mathbf{0}$$

(only the impossible event is orthogonal to itself).

7. *Orthomodularity*:

$$a \leq b \Rightarrow \exists c\ [a \perp c \text{ and } \sup(a, c) = b].$$

8. *Weak determinism* (S is *suffcient* and E is *unital*):

$$a \neq \mathbf{0} \Rightarrow \exists u\ [u(a) = \mathbf{1}]$$

(any event which is not impossible is satisfied with certainty by at least one state).

9. The order is determined by the certainty domains

$$\{u/u(a) = 1\} \subseteq \{u/u(b) = 1\} \Rightarrow a \leq b$$

10. The event-poset is a lattice, with other possible "nice" properties, like σ-completeness, completeness, atomicity, the covering property, distributivity, and so on.

11. The set of the states is σ-convex.

As is well known, the orthodox *sharp* approach to QT and the more general *unsharp* approach (Busch, 1985; Busch *et al.*, 1991; Cattaneo, 1993; Cattaneo and Laudisa, n.d.; Davies, 1976; Kraus, 1983; Ludwig, 1983) have given different answers to the question concerning the validity of our conditions. In the orthodox sharp approach, events give rise to a structure that is at least a σ-orthomodular poset, which is often termed simply a *quantum logic*. Further, $\langle S, E \rangle$ satisfies the σ-additivity, the weak determinism condition; and the event-order is determined by the certainty domains. In the framework of the unsharp approach, instead, events give rise only to a bounded regular poset, where the noncontradiction principle and the orthomodular property are generally violated. A partial sum $\oplus$ is defined (for pairs of orthogonal events); however, $a \oplus b$ is not, generally, the supremum of a and b.

Canonical Hilbert-space exemplifications of both the sharp and the unsharp state-event systems can be constructed by taking, respectively, as events either the *projectors* or the *effects* of the Hilbert space $\mathscr{H}$ (associated with the physical system under investigation). At the same time, states are identified with the density operators of $\mathscr{H}$ in both approaches.

The advantages of the unsharp approach have been illustrated in a large literature. From a general point of view, one could say that moving to the unsharp approach represents an important step toward a kind of "second degree of fuzziness." Namely, in the framework of the sharp approach, any event a can be regarded as a kind of "clear" property. Whenever a state u assigns to a a probability value different from 1 and 0, one can think that the semantic uncertainty involved in such a situation totally depends on the ambiguity of the state and not on the ambiguity of the property (first degree of fuzziness). In the unsharp approach, instead, one takes into account also "genuine ambiguous properties;" an extreme case is represented by the *semitransparent property* **1/2**, to which any state assigns probability 1/2. This second degree of fuzziness may be regarded as depending on the accuracy of the measurement (which triggers the property), and also on the accuracy involved in the operational definitions for the physical quantities to which our property refers.

2. SHARP AND UNSHARP QUANTUM LOGICS

Let us now turn to the question concerning the logics that can be associated to collections of state-event systems. In the logical tradition, generally a logic **L** can be characterized by means of two privileged kinds of semantics:

1. An *algebraic semantics*.
2. A *possible world semantics* (called also *Kripke semantics*).

These semantics give different answers to the question, "What does it mean to interpret a formal language?"

In the algebraic approach, the basic idea is that interpreting a language essentially means associating to any *sentence* of the language an abstract *truth-value* or more generally an abstract *meaning*, identified as an element of a convenient algebraic structure. Hence, generally, an *algebraic model* for a logic **L** will have the form

$$\mathcal{M} = \langle \mathcal{A}, v \rangle$$

where $\mathcal{A}$ is an abstract structure belonging to a class **K** of structures (satisfying the same set of conditions) and v transforms the sentences into elements of $\mathcal{A}$, preserving the logical form (in other words, logical connectives are interpreted as operations of the structure). We require that a partial order relation $\leq$ is defined in any $\mathcal{A}$. The *truths* of the model are the sentences that receive value **1**.

The semantic consequence relation is then defined as follows:

Definition 2.1. $T \models \alpha$ (α is a *semantic consequence* of T) iff for any model $\mathcal{M} = \langle \mathcal{A}, v \rangle$ and any element a of $\mathcal{A}$

$$\text{if for any sentence } \beta \text{ of } T, \quad a \leq v(\beta), \quad \text{then } a \leq v(\alpha)$$

In the possible world semantics, instead, one assumes that interpreting a language essentially means associating to any sentence α the set of the *possible worlds* (or *situations*) where the sentence α certainly holds. This set, which represents the *extensional meaning* of α, is usually called the *proposition* associated to α (or simply *the proposition of* α).

Hence, generally, a Kripke model will have the form

$$\mathcal{M} = \langle I, R_1, \ldots, R_m, o_1, \ldots, o_n, \Pi, v \rangle$$

where:

1. I is a nonempty set of possible worlds, possibly correlated by relations R_i and operations o_j. In most cases, we have only one relation, called the *accessibility relation*.
2. Π is a set of sets of possible worlds, representing possible propositions of the sentences. Any single proposition and the total set Π must satisfy convenient closure conditions that depend on the particular logic.
3. v (the *interpretation function*) transforms any sentence into a proposition in Π, preserving the logical form.

Given a model $\mathcal{M}$ and a world i of $\mathcal{M}$, i is said to *verify* a sentence α ($i \models \alpha$) iff i belongs to the proposition $v(\alpha)$. The *truths* of the model are the sentences that are verified by any world.

Any class **K** of all models $\mathcal{M}$ satisfying the same set of conditions determines a semantic consequence relation defined as follows:

Definition 2.2. $T \vDash \alpha$ (α is a *semantic consequence* of T) iff for any model $\mathcal{M}$ and any world i of $\mathcal{M}$

if i verifies all the sentences in T, then i verifies also α

It is interesting to consider also a variant of Kripke semantics, which was first applied to QT by Bugajski, Cattaneo, and Nisticò. We might call this approach a *many-valued possible world semantics*. The basic idea is a generalization of the notion of *proposition*. As we have seen, in the standard Kripke semantics the proposition of a sentence α is a set of worlds: the worlds where α certainly holds. This automatically determines the set of the worlds where α certainly does not hold (in other words, the meaning of the negation of α). Intermediate truth-values are not considered. In the many-valued possible world semantics, instead, one fixes, at the very beginning, a set of truth-values $W \subseteq [0, 1]$ and any proposition is represented as a function which associates to any truth-value in W a convenient set of worlds (the worlds where our sentence holds with that particular truth-value). As a consequence, the total set of propositions Π turns out to behave like a family of fuzzy sets of worlds.

State-event systems can be transformed both into algebraic and Kripkean models in a natural way. Given $\langle S, E \rangle$, an algebraic model $\langle \mathcal{A}, v \rangle$ can be constructed by taking as $\mathcal{A}$ the algebraic structure of the events and by assuming that the interpretation function v follows the intended physical meaning of the atomic sentences. At the same time, a Kripke model $\langle I, R, \Pi, v \rangle$ can be obtained by identifying the set of worlds I with the set of the states and by assuming that two states u and v are accessible (Ruv) iff they are not *strongly discernible* (there exists no event a such that $u(a) = 1$ and $v(a) = 0$); further, the set Π of the propositions is determined by the set of events, and the interpretation function v follows the intended physical meaning of the atomic sentences.

Alternatively, a Kripke model can be obtained also by identifying the possible worlds with the events, and the accessibility relation with the nonorthogonality relation. From an intuitive point of view, in this case, our worlds will represent possible descriptions of physical systems, which are generally only partial.

What about the quantum logics that can be generated by these methods? It seems natural to imagine the "quantum logical population" as a kind of "Ptolemaic universe" whose center is occupied by Birkhoff–von Neumann *orthodox quantum logic* (QL), in the same way as the total logical universe has in its center classical logic.

QL represents a singularity in the class of all logics. Many metalogical problems have been solved. However, some questions seem to be stubbornly resistant to any attempt of solution. Among the solved problems let us mention at least the following results:

(a) QL can be characterized both in the algebraic and in the Kripke semantics. Algebraically, it is characterized by the class of all models $\langle \mathscr{A}, v \rangle$, where $\mathscr{A}$ is an orthomodular lattice and v interprets the logical connectives in the expected way:

$$v(\neg\beta) = v(\beta)^{\perp}; \qquad v(\beta \wedge \gamma) = \inf(\{\beta, \gamma\})$$

(where $^{\perp}$ and inf are, respectively, the orthocomplement and the infimum in $\mathscr{A}$).

In the Kripke semantics, QL is characterized by the class of all models

$$\langle I, R, \Pi, v \rangle$$

where:

1. R is a reflexive and symmetrical relation. A *possible proposition* of the model is a *maximal* set of worlds X, which contains all and only the worlds i, whose accessible worlds are accessible to at least one element of X. In other words

$i \in X$ iff for any j accessible to i there exists a k accessible to j such that $k \in X$

For any set of worlds X, let $X^{\perp} = \{i \in I / \forall j \text{ (if } Rij, \text{ then } j \notin X)\}$.

2. Π is a set of possible propositions such that
2.1. Π contains $\varnothing$, I; and is closed under the set-theoretic intersection and under the operation $^{\perp}$.
2.2. Π satisfies the orthomodular property:

$$X \cap (X \cap (X \cap Y)^{\perp})^{\perp} \subseteq Y$$

3. $v(\neg\beta) = v(\beta)^{\perp}$; $v(\beta \wedge \gamma) = v(\beta) \cap v(\gamma)$.

As proved by Goldblatt (1984), the orthomodularity of the set of propositions cannot be expressed as an elementary property of the accessibility relation.

(b) QL is axiomatizable. Many axiomatizations are known: in the *logicistic* style, as a set of rules, in the *natural deduction* style, in the *sequent* style (e.g., Dalla Chiara, 1986; Dishkant, 1972; Gibbins, 1987; Goldblatt, 1974; Nishimura, 1980).

(c) QL is not characterized by the class of all algebraic models based on a *Hilbert lattice* (the lattice of the projectors in a separable complex

Hilbert space).[3] Hence QL is definitely more general with respect to its historical and physical origin.

Among the questions that are still unsolved, let us mention at least the following:

(a) Is QL decidable?

(b) Does QL admit the *finite model property*? In other words, if a sentence is not a quantum logical truth, is there any finite model where our sentence is not true? A positive answer to the finite model property would automatically provide a positive answer to the decidability question, but not vice versa.

(c) Is the set of all possible propositions in the Kripke *canonical* model of QL orthomodular? (The worlds of the canonical model are all the noncontradictory and deductive closed sets of sentences T; whereas two worlds T and T' are accessible iff whenever T contains a sentence α, T' does not contain its negation $\neg\alpha$). This problem is correlated to the critical question of whether any orthomodular lattice is embeddable into a complete orthomodular lattice. Only partial answers are known.

(d) Is the logic characterized by all Hilbert lattices axiomatizable?

By dropping the orthomodular condition (both in the algebraic and the Kripke semantics), one may obtain weaker forms of quantum logic that turn out to be more tractable from a metalogical point of view. Two examples are represented respectively by *minimal quantum logic* (MQL) (called also *orthologic*), which is characterized by the class of all algebraic models based on ortholattices, and *paraconsistent quantum logic* (PQL), characterized by the class of all algebraic models based on involutive bounded lattices (possibly violating the noncontradiction and the excluded middle principles). Both logics satisfy the finite model property. PQL is a common sublogic of the *Brouwer–Zadeh logics* [which represent natural logical abstractions from the unsharp approach to QT (Cattaneo, 1993; Giuntini, 1993)] and of Łukasiewicz infinite many-valued logic [whose application to QT has been studied by Mundici and in the fuzzy quantum logical approaches (Mundici, 1993; Pykacz, 1993)].

So far we have considered only examples of quantum logics where conjunctions and disjunctions are supposed to be always defined. However, the experimental and the probabilistic meaning of conjunctions of incompatible propositions in QT has been often put in question. How to construct logics where we admit that conjunctions and disjunctions are possibly meaningless? For instance, how to give a natural semantic characterization for a logic corresponding to the class of all orthomodular posets or to the class of all orthoalgebras? Let us call these logics, respectively,

[3] This has been proved by Greechie (1971) and Kalmbach (1974).

strong partial quantum logic (SPaQL) and *weak partial quantum logic* (WPaQL). Are SPaQL and WPaQL axiomatizable? The question admits a positive answer (Foulis *et al.*, n.d.). An axiomatizable logic that turns out to be slightly stronger than SPaQL is *transitive partial classical logic* (TPaCL), characterized by the class of all models based on transitive partial Boolean algebras. TPaCL is, of course, stronger than *partial classical logic* (PaCL), characterized by the class of all models based on partial Boolean algebras. At the same time, PaCL and the partial quantum logics turn out to be uncomparable.

3. CONCLUSIONS

Some general questions that have been often discussed in connection with (or against) quantum logic are the following:

(a) Why quantum logics?

(b) Are quantum logics helpful to solve the difficulties of QT?

(c) Are quantum logics "real logics"? And how is their use compatible with the mathematical formalism of QT, based on classical logic?

My answer to these questions is, in a sense, trivial [and close to a position that Gibbins (1991) has defined "a quietist view of quantum logic"].

It seems to me that quantum logics are not to be regarded as a kind of "clue," capable of solving the main physical and epistemological difficulties of QT. This was perhaps an illusion of some pioneering workers in quantum logic. Let us think of the attempts to recover a *realistic interpretation* of QT based on the properties of the quantum logical connectives.

Why quantum logics? Simply because "quantum logics are there!" They seem to be deeply incorporated in the abstract structures generated by QT. Quantum logics are, without any doubt, *logics*. For, they satisfy all the canonical conditions that the present community of logicians require in order to call a given abstract object a *logic*. The compresence of different logics in one and the same theory may give, *prima facie*, a feeling of uneasiness. However, the splitting of the traditional connectives (*not*, *and*, *or*, . . .) into different logical constants, with different meanings and uses, is today a well-accepted logical phenomenon, which is in no way specific of QT.

In this paper, I have considered only quantum logics at the sentential level. From a strictly logical point of view, first-order extensions of some quantum logics (for instance, orthodox quantum logic) are not problematic. However, formidable problems arise at the interpretation level when we admit that the domains of our first-order models contain physical objects. This leads to hard questions, concerning, for instance, the concepts of *identity* and *genidentity*, the logical and ontological status of *virtual particles*, and so on. A lot of interesting semantic analysis has been done in this field, according to more or less formal methods (e.g., van Fraassen, 1991; Krause, 1992; Castellani and Mittelstaedt, n.d.; Dalla Chiara and Toraldo di Francia, 1993). However, we are still far from definite results, and much further work will probably be required.

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